WHEN IS THE HAAR MEASURE A PIETSCH MEASURE? - PART 1
GDT - AF

1. Introduction

Absolutely p-summing linear operators. These operators improve, in a certain sense, the notion of summability of sequences.

- Grothendieck (1956): Absolutely 1-*summing linear operators* (using a tensor language).
- Pietsch, Mitiagin, Pelczyński and Lindenstrauss (60'): Absolutely p-*summing linear operators* (using a language closer to the language currently used, also showed the first fundamental properties of these operators).

Let F and X be a Banach spaces, and $1 \leq p < \infty$. A continuous linear operator $u : F \to X$ is *absolutely p-summing* if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $T_1, \ldots, T_n \in F$, we have

$$\left(\sum_{j=1}^{n} \|u(T_j)\|^p \right)^{1/p} \leq C \sup_{\varphi \in B_{F^*}} \left(\sum_{j=1}^{n} |\varphi(T_j)|^p \right)^{1/p},$$

where B_{F^*} is the unit closed ball of the topologic dual F^* of F.

Theorem 1 (Pietsch's Domination Theorem (PDT))**.** An operator $u : F \to X$ is absolutely p-summing if and only if there are a constant $C > 0$ and a Borel probability regular measure μ on $(B_{F^*}, \sigma(F^*, F))$ such that for all $T \in F$, we have

$$\|u(T)\| \leq C \left(\int_{B_{F^*}} |\varphi(T)|^p \, d\mu(\varphi) \right)^{1/p}. \tag{1}$$

The infimum of all constants $C > 0$ thats satisfies (1) is denoted by $\pi_p(u)$ and also satisfies the inequality. A measure μ thats satisfies (1) is called *Pietsch measure* for u.

Remark 2. When G to be a compact Hausdorff topological group, a reformulation of the PDT tells us that $u : C(G) \to X$ is absolutely p-summing if and only if there is a Borel probability measure μ on G such that, for all $T \in C(G)$, we have

$$\|u(T)\| \leq \pi_p(\mu) \left(\int_{G} |T(\varphi)|^p \, d\mu(\varphi) \right)^{1/p}. \tag{2}$$

A measure μ on G thats satisfies (2) also is called Pietsch measure.

In general we little can be said about the measures which appear in the PDT, because the Pietsch measure comes into the proof from abstract existence results: Hahn-Banach and Riesz representation theorems for the dual of $C(G)$-spaces. However, under certain conditions, a Pietsch measure is known for the absolutely p-summing operators.

Theorem 3. Let X be a Banach space, G be a compact Hausdorff topological group, F be a closed *translation invariant* (see Definition 6(i)) subspace of $C(G)$, and $u : F \to X$ a translation invariant absolutely p-summing operator. Then the normalized *Haar measure* (see Definition 7) σ_G on G is a Pietsch measure.

The proof of this Theorem can be found in Diestel's book, p. 56.

Nonlinear generalizations of p-summing linear operators. Aims to extend the concept of absolutely p-summing linear operator to a nonlinear environment.

- Pietsch and Alencar-Matos (80'): Multilinear and polynomial environment.
- Farmer and Johnson (2009): Lipschitz environment.

Here, we will focus on the Lipschitz maps.

Let (F, d_F, e_F) be a pointed metric space, (X, d_X) be a metric space, and $1 \leq p < \infty$. A map $u : F \to X$ is *Lipschitz p-summing* if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $T_1, \ldots, T_n, S_1, \ldots, S_n \in F$, we have

$$\left(\sum_{j=1}^{n} d_X(u(T_j), u(S_j))^p \right)^{1/p} \leq C \sup_{\varphi \in B_{F^\#}} \left(\sum_{j=1}^{n} |\varphi(T_j) - \varphi(S_j)|^p \right)^{1/p},$$

where $F^\#$ is the *Lipschitz dual* of F, that is, $F^\#$ is the Banach space of all Lipschitz functions $\varphi : F \to \mathbb{K}$ such that $\varphi(e_F) = 0$, endowed with the norm:

$$\|\varphi\|_{F^\#} := \sup_{\substack{T,S \in F \\ T \neq S}} \frac{|\varphi(T) - \varphi(S)|}{d_F(T, S)}.$$

Theorem 4. A map $u : F \to X$ is Lipschitz p-summing if there are a constant $C > 0$ and a Borel probability regular measure μ on $B_{F^\#}$ such that for all $T, S \in F$, we have

$$d_X(u(T), u(S)) \leq C \left(\int_{B_{F^\#}} |\varphi(T) - \varphi(S)|^p \, d\mu(\varphi) \right)^{1/p}.$$

Diestel, Jarchow and Tonge posed the following problem: Is the Haar measure a Pietsch measure in more general contexts (as for example for Lipschitz p-summings maps)?

Generalization of p-summing linear operators in a abstract context. Several others classes of operators are be defined in a similar way to the absolutely p-summing operators. Hence:

- G. Botelho, D. Pellegrino and P. Rueda (2010): Abstract environment (introduced an abstract class of operators, unifying the most varied classes of operators, and provided an abstract version of PDT).
- D. Pellegrino and J. Santos (2011): bstract environment (improved the abstract version of the PDT, requiring fewer hypotheses).

Let X, Y and F be an arbitrary sets, $\mathcal{H}$ be an arbitrary family of maps from Y to X, K be a compact Hausdorff topological space, Z a Banach space, $0 < p < \infty$, and $R : K \times F \times Z \to [0, \infty)$

and $S : \mathcal{H} \times F \times Z \to [0, \infty)$ be a arbitrary maps. An map $u \in \mathcal{H}$ is *R-S-abstract p-summing* if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in F$, $b_1, \ldots, b_n \in G$, we have

$$\left(\sum_{j=1}^{n} S(u, T_j, b_j)^p \right)^{1/p} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^{n} R(\varphi, T_j, b_j)^p \right)^{1/p}.$$

Theorem 5 (Pietsch's Domination Theorem Unified (PDTU))**.** Suppose that $R(\cdot, T, b) : K \to [0, \infty)$ is a continuous map, for all $T \in F$ and $b \in Z$. Then a map $u \in \mathcal{H}$ is R-S-abstract p-summing if and only if there are a constant $C > 0$ and a Borel probability regular measure μ on K such that for all $T \in F$ and $b \in G$, we have

$$S(u, T, b) \leq C \left(\int_K R(\varphi, T, b)^p \, d\mu(\varphi) \right)^{1/p}. \tag{3}$$

The infimum of all constants $C > 0$ thats satisfies (3) is denoted by $\pi_{RS,p}(u)$ and also satisfies the inequality. The measure μ thats satisfies (3) is called *R-S-abstract measure* for u.

The main result provide us the validity of Theorem 3 in the abstract context of the PDTU.

2. Preliminaries and background

Let G be a compact Hausdorff topological group. We will denote by $C(G)$ the Banach space of all continuous functions $T : G \to \mathbb{K}$ endowed with the sup norm:

$$\|T\|_{C(G)} := \sup_{\varphi \in G} |T(\varphi)|$$

(here it is sufficient that G to be a Hausdorff compact topological space). Given $T \in C(G)$ and $\psi \in G$, we will define the following function:

$$T^{\psi} : G \to \mathbb{R}$$
$$\varphi \mapsto T^{\psi}(\varphi) := T(\psi\varphi).$$

Clearly $T^{\psi} \in C(G)$ (here we need that G to be a topological group). Furthermore, we have

$$\|T^{\psi}\| = \sup_{\varphi \in G} |T^{\psi}(\varphi)| = \sup_{\varphi \in G} |T(\psi\varphi)| = \sup_{\varphi \in G} |T(\varphi)| = \|T\|.$$

Definition 6. Let G be a compact Hausdorff topological group.

- (i) A noemptyset $F \subseteq C(G)$ is *translation invariant* if for all $T \in F$ and $\psi \in G$, we have $T^{\psi} \in F$.
- (ii) Let X be an arbitrary set and F be a closed translation invariant subspace of $C(G)$. A map $u : F \to X$ is *translation invariant* if for each $T \in F$ and $\psi \in G$, we have $u(T^{\psi}) = u(T)$.
- (iii) A measure σ on G is *translation invariant* if for all Borel subset $B \subseteq G$ and $\psi \in G$, we have $\sigma(\psi B) = \sigma(B)$.

Definition 7. A *Haar measure* on a compact Hausdorff topological group G is a Radon measure which is translation invariant.

Remark 8. There is a unique normalized Haar measure on compact Hausdorff topological group G. Let us denote such a measure by σ_G.

Before proceeding to the result, we will remember of Riesz's Representation Theorem for the dual of $C(G)$.

Theorem 9 (Riesz's Representation Theorem)**.** Let G be a compact Hausdorff topological group. A functional $f \in C(G)^*$ is positive if and only if there is a Radon measure μ on G such that, for all $T \in C(G)$, we have

$$f(T) = \int_G T(\varphi)\, d\mu(\varphi).$$

The proof of this Theorem can be found in Ash's book, Proposition 4.3.10.

Denoting by $1_G : G \to \mathbb{K}$ the function $\varphi \in G \mapsto 1 \in \mathbb{K}$, we have

$$\mu(G) = \int_G 1\, d\mu(\varphi) = \int_G 1_G(\varphi)\, d\mu(\varphi) = f(1_G).$$

Thus, f is a translation invariant functional if and only if for all $T \in C(G)$ and $\psi \in G$, we have

$$f(T) = \int_G T(\varphi)\, d\mu(\varphi) = \int_G T(\psi\varphi)\, d\mu(\varphi) = f(T^\psi).$$

3. Main result

Theorem 10. Let X and Y be an arbitrary sets, $\mathcal{H}$ be an arbitrary family of maps from Y to X, $u \in \mathcal{H}$, G be a compact Hausdorff topological group, F be a closed translation invariant subspace of $C(G)$, Z be a Banach space, $0 < p < \infty$, and $R : G \times F \times Z \to [0, \infty)$ and $S : \mathcal{H} \times F \times Z \to [0, \infty)$ be an arbitrary maps satisfying the following conditions:

(i) $R(\cdot, T, b) : G \to [0, \infty)$ is a continuous map, for all $T \in F$ and $b \in Z$.
(ii) $R(\varphi, T^\psi, b) \leq R(\psi\varphi, T, b)$, for all $\varphi, \psi \in G$, $T \in F$ and $b \in Z$.
(iii) $S(u, \cdot, b) : F \to [0, \infty)$ is translation invariant, for all $b \in Z$.

If u is R-S-abstract p-summing, then the normalized Haar measure σ_G on G is R-S-abstract measure for u.

Proof. By PDTU, there is a R-S-abstract measure μ on G for u. Let $f \in C(G)^*$ the positive functional associeted with μ. For each $\psi \in G$, defines

$$\begin{aligned} f_\psi : C(G) &\to \mathbb{K} \\ T &\mapsto f_\psi(T) := f(T^\psi). \end{aligned}$$

We can proof that $f_\psi \in C(G)^*$ is a positive functional. Then, consider μ_ψ the Radon measure associeted with f_ψ. Note that μ_ψ is a probabilidade measure, since

$$\mu_\psi(G) = f_\psi(1_G) = f(1_G^\psi) = \int_G 1_G^\psi(\varphi)\, d\mu(\varphi) = \int_G 1_G(\psi\varphi)\, d\mu(\varphi) = \int_G 1\, d\mu(\varphi) = \mu(G) = 1.$$

Furthermore, note that μ_ψ is an R-S-abstract measure for u, since given $T \in F$ and $b \in Z$, we have

$$\begin{aligned} S(u,T,b) &= S(u,\cdot,b)(T) = S(u,\cdot,b)(T^\psi) = S(u,T^\psi,b) \\ &\leq \pi_{RS,p}(u)\left(\int_G R(\varphi,T^\psi,b)^p\,d\mu(\varphi)\right)^{1/p} \leq \pi_{RS,p}(u)\left(\int_G R(\psi\varphi,T,b)^p\,d\mu(\varphi)\right)^{1/p} \\ &= \pi_{RS,p}(u)\left(\int_G R(\cdot,T,b)^p(\psi\varphi)\,d\mu(\varphi)\right)^{1/p} = \pi_{RS,p}(u)\left(\int_G [R(\cdot,T,b)^p]^\psi(\varphi)\,d\mu(\varphi)\right)^{1/p} \\ &= \pi_{RS,p}(u) f([R(\cdot,T,b)^p]^\psi)^{1/p} = \pi_{RS,p}(u) f_\psi(R(\cdot,T,b)^p)^{1/p} \\ &= \pi_{RS,p}(u)\left(\int_G R(\cdot,T,b)^p(\varphi)\,d\mu_\psi(\varphi)\right)^{1/p} = \pi_{RS,p}(u)\left(\int_G R(\varphi,T,b)^p\,d\mu_\psi(\varphi)\right)^{1/p}. \end{aligned}$$

Now, we claim that the map

$$I_\mu : G \to C(G)^*$$
$$\psi \mapsto I_\mu(\psi) := f_\psi$$

is continuous, when $C(G)^*$ it is endowed with the weak* topology. To check this, for each $T \in C(G)$, we will show that $J_{C(G)}(T) \circ I_\mu : G \to \mathbb{K}$ is a continuous map, where $J_{C(G)} : C(G) \to C(G)^{**}$ is the canonical embedding of $C(G)$. Indeed, given $T \in C(G)$ and $\psi \in G$, we have

$$J_{C(G)}(T) \circ I_\mu(\psi) = J_{C(G)}(T)(f_\psi) = f_\psi(T) = f(T^\psi) = \int_G T^\psi(\varphi)\,d\mu(\varphi) = \int_G T(\psi\varphi)\,d\mu(\varphi),$$

which is continuous. Consider now the normalized Haar measure σ_G on G. Then there is $h \in C(G)^*$ such that, for all $T \in C(G)$, we have

$$h(T) = \int_G f_\phi(T)\,d\sigma_G(\phi) = \int_G f(T^\phi)\,d\sigma_G(\phi) \geq 0$$

(see p.57 of Diestel's book). Note that h is a positive functional, since f is positive. It follows that there is Radon measure ν on G such that

$$h(T) = \int_G T(\phi)\,d\nu(\phi).$$

Moreover, note that ν is a probability measure, since

$$\nu(G) = h(1_G) = \int_G f_\phi(1_G)\,d\sigma_G(\phi) = \int_G \mu_\phi(G)\,d\sigma_G(\phi) = \int_G 1\,d\sigma_G(\phi) = \sigma_G(G) = 1.$$

Furthermore, note that ν is an R-S-abstract measure for u, since given $T \in F$ and $b \in Z$, we have

$$\begin{aligned} \frac{S(u,T,b)^p}{\pi_{RS,p}(u)^p} &= \frac{S(u,T,b)^p}{\pi_{RS,p}(u)^p}\int_G 1\,d\sigma_G(\phi) = \int_G \frac{S(u,T,b)^p}{\pi_{RS,p}(u)^p}\,d\sigma_G(\phi) \leq \int_G f_\phi(R(\cdot,T,b)^p)\,d\sigma_G(\phi) \\ &= h(R(\cdot,T,b)^p) = \int_G R(\cdot,T,b)^p(\phi)\,d\nu(\phi) = \int_G R(\phi,T,b)^p\,d\nu(\phi). \end{aligned}$$

We will show now that ν is a translation invariant measure. For this, is enough to show that h is a translation invariant functional. Indeed, given any $T \in C(G)$ and $\psi \in G$, we have

$$\begin{aligned} h(T^{\psi}) &= \int_G f_{\phi}(T^{\psi})\, d\sigma_G(\phi) = \int_G f((T^{\psi})^{\phi})\, d\sigma_G(\phi) = \int_G \int_G (T^{\psi})^{\phi}(\varphi)\, d\mu(\varphi)\, d\sigma_G(\phi) \\ &= \int_G \int_G T(\psi\phi\varphi)\, d\mu(\varphi)\, d\sigma_G(\phi) = \int_G \int_G T(\psi\phi\varphi)\, d\sigma_G(\phi)\, d\mu(\varphi) = \int_G \int_G T(\phi\varphi)\, d\sigma_G(\phi)\, d\mu(\varphi) \\ &= \int_G \int_G T(\phi\varphi)\, d\mu(\varphi)\, d\sigma_G(\phi) = \int_G \int_G T^{\phi}(\varphi)\, d\mu(\varphi)\, d\sigma_G(\phi) = \int_G f(T^{\phi})\, d\sigma_G(\phi) \\ &= \int_G f_{\phi}(T)\, d\sigma_G(\phi) = h(T). \end{aligned}$$

It follows that ν is a normalized Haar measure on G. By unicity of the normalized Haar measure, we conclude that $\nu = \sigma_G$. □

References

[1] R. Ash, *Measure, Integration and Functional Analysis*, Academic Press, Inc., 1972.

[2] G. Botelho, D. Pellegrino and P. Rueda, *A unified Pietsch domination theorem*, J. Math. Anal. Appl. 365 (2010), 269–276.

[3] G. Botelho, D. Pellegrino, P. Rueda, J. Santos and J. B. Seoane-Sepúlveda, *When is the Haar measure a Pietsch measure for nonlinear mappings?*, Studia Mathematica 213 (2012), 275–287.

[4] G. Botelho, D. Pellegrino, E. Teixeira, *Fundamentos de Análise Funcional*, Sociedade Brasileira de Matemática, 2015.

[5] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge Univ. Press., Cambridge, 1995.

[6] J. Farmer and W. B. Johnson *Lipschitz p-summing operators*, Proc. Amer. Math. Soc. 137 (2009), 2989–2995.

[7] D. Pellegrino and J. Santos, *A general Pietsch domination theorem*, J. Math. Anal. Appl. 375 (2011), 371–374.

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