Common hypercyclicity in several dimensions

Analysis and Geometry Seminar

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Hypercyclicity and common hypercyclicity

- Let X be a Fréchet space and $T : X \to X$ be linear and continuous (a.k.a. operator).

We say that $x \in X$ is a **hypercyclic vector** for T when its orbit $Orb(x, T) := \{T^n x : n \geq 0\}$ is dense in X. If such a vector exists we say that T is a **hypercyclic operator** on X. The set of hypercyclic vectors for an operator T is denoted by $HC(T)$.

- Let $d \geq 1$, let $\Lambda \subset \mathbb{R}^d$ be σ-compact and let $(T_\lambda)_{\lambda\in\Lambda}$ be a family of operators acting on X such that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous (a.k.a. continuous family).

We say that $x \in X$ is a **common hypercyclic vector** for $(T_\lambda)_{\lambda\in\Lambda}$ when $x \in HC(T_\lambda)$ for all $\lambda \in \Lambda$. If such a vector exists we say that $(T_\lambda)_{\lambda\in\Lambda}$ is a **common hypercyclic family** on X.

First common hypercyclicity results ($d = 1, 2$)

Theorem (Abakoumov and Gordon (2003))

For $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$, the family $(\lambda B)_{\lambda>1}$ has a common hypercyclic vector. (Even a G_δ-dense set of them, actually!)

Borichev's example

For any $\Lambda \subset (1, +\infty) \times (1, +\infty)$ with positive Lebesgue measure, $(\lambda B \times \mu B)_{(\lambda,\mu)\in\Lambda}$ has no common hypercyclic vector on $\ell_1(\mathbb{N}) \times \ell_1(\mathbb{N})$.

- Let $\mathcal{D}$ be a dense subset of X on which each T_λ has a right inverse $S_\lambda : \mathcal{D} \to X$.

Example

Let $X = \ell_1(\mathbb{N})$ and $\big(w(\lambda)\big)_{\lambda\in\Lambda}$ be a family of admissible weights. On the dense subset $\mathcal{D} = c_{00}(\mathbb{N})$ of $\ell_1(\mathbb{N})$, each $T_\lambda := B_{w(\lambda)}$, $\lambda \in \Lambda$, has the forward shift $F_{w(\lambda)^{-1}}$ for the inverse weight $w(\lambda)^{-1} = \big(\frac{1}{w_n(\lambda)}\big)_n$ as a well-defined right inverse.

First common hypercyclicity results ($d = 1$)

Theorem (Costakis and Sambarino (2004))

Assume that, for each $v \in \mathcal{D}$ and each compact interval $K \subset \Lambda$, the following properties hold true, where all parameters α, λ, μ belong to K.

(1) *There exist $\kappa \in \mathbb{N}$ and a sequence of positive numbers $(c_k)_{k \geq \kappa}$ such that*
- $\displaystyle\sum_{k=\kappa}^{\infty} c_k < \infty$;
- $\|T_\lambda^{n+k} S_\alpha^n(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\alpha \leq \lambda$;
- $\|T_\lambda^{n} S_\alpha^{n+k}(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\lambda \leq \alpha$

(2) *Given $\eta > 0$, one can find $\tau > 0$ such that, for all $n \geq 1$,*

$$0 \leq \mu - \lambda < \frac{\tau}{n} \implies \|T_\lambda^n S_\mu^n(v) - v\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ subset of X.

First common hypercyclicity results ($d \geq 1$)

Theorem (Basic criterion [3])

Assume the following. For each compact $K \subset \Lambda$, each pair $u, v \in \mathcal{D}$ and each open neighborhood O of the origing, one can find parameters $\lambda_1, ..., \lambda_q \in \Lambda$, sets of parameters $\Lambda_1, ..., \Lambda_q \subset \Lambda$, with $\lambda_i \in \Lambda_i, i = 1, ..., q$, and integers $n_1, ..., n_q \in \mathbb{N}$, such that

(i) $\displaystyle\bigcup_{i=1}^{q} \Lambda_i \supset K$;

(ii) $\forall i = 1, ..., q, \forall \lambda \in \Lambda_i : \displaystyle\sum_{j=1}^{q} S_{\lambda_j}^{n_j}(v) \in O$ *and* $\sum_{j \neq i} T_{\lambda}^{n_i} S_{\lambda_j}^{n_j}(v) \in O$;

(iii) $\forall i = 1, ..., q, \forall \lambda \in \Lambda_i : T_{\lambda}^{n_i}(u) \in O$

(iv) $\forall i = 1, ..., q, \forall \lambda \in \Lambda_i : T_{\lambda}^{n_i} S_{\lambda_i}^{n_i}(u) - u \in O$.

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ *is a* G_δ *subset of* X.

Understanding the basic criterion with an example

Example ($d = 1$)

Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$, let K be a compact interval and let $\big(w(x)\big)_{x\in K}$ be a positive family of admissible weights such that the map $x \mapsto w_n(x)$ is increasing for all $n \in \mathbb{N}$. Assume the following:

- $\sum_{n=1}^{\infty} \frac{1}{w_1(x)\cdots w_n(x)} < +\infty$;
- for all $n \in \mathbb{N}$, the function $x \overset{f_n}{\longmapsto} \sum_{k=1}^{n} \log\big(w_k(x)\big)$ is n-Lipschitz.

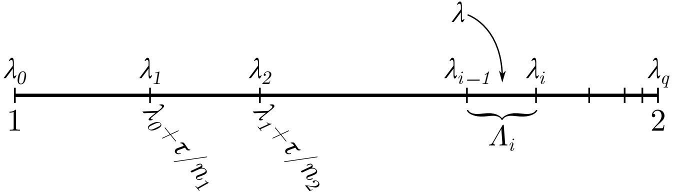
Then $\big(B_{w(x)}\big)_{x\in K}$ has a common hypercyclic vector.

Sketch of proof.

We consider $K = [1, 2]$ and $u = v = e_0 = (1, 0, 0, ...)$. We want to define a tagged partition $(\lambda_i, \Lambda_i)_{i=1,\ldots,q}$ of K and associate powers $n_1, ..., n_q$ to each part.

Understanding the basic criterion with an example

Let us consider the partition as in the following figure.



For this to work, we need

$$\sum_{i=1}^{q} \frac{\tau}{n_i} \geq 2 - 1 = 1. \tag{1}$$

This will not be a problem, as we will choose $(n_i)_i$ such that $\sum_{i=1}^{\infty} \frac{1}{n_i} = +\infty$. This way we can construct the partition even if τ is small. The size of q will increase, but this is not a problem here.
By doing this, condition (i) of the basic criterion is satisfied.

Understanding the basic criterion with an example

For condition (ii) of the Basic Criterion, we first look at the sum

$$\sum_{j=1}^{q} S_{\lambda_j}^{n_j}(e_0) = \sum_{j=1}^{q} F_{w(\lambda_j)^{-1}}^{n_j}(e_0) = \sum_{j=1}^{q} \frac{1}{w_1(\lambda_j)\cdots w_{n_j}(\lambda_j)} e_{n_j}$$

From the hypothesis, this is small as long as we choose n_1 big enough. This can be easily fulfilled as we plan to define $n_k = kN$ for some big N.

Understanding the basic criterion with an example

For the second condition in (ii), we take $\lambda \in \Lambda_i$ for some $i \in \{1, ..., q\}$ and calculate

$$\begin{aligned}
\sum_{j \neq i} T_\lambda^{n_i} S_{\lambda_j}^{n_j}(e_0) &= \sum_{j=i+1}^{q} B_{w(\lambda)}^{n_i} F_{w(\lambda_j)^{-1}}^{n_j}(e_0) \\
&= \sum_{j=i+1}^{q} B_{w(\lambda)}^{n_i} \frac{1}{w_1(\lambda_j) \cdots w_{n_j}(\lambda_j)} e_{n_j} \\
&= \sum_{j=i+1}^{q} \frac{w_{n_j - n_i + 1}(\lambda) \cdots w_{n_j}(\lambda)}{w_1(\lambda_j) \cdots w_{n_j}(\lambda_j)} e_{n_j - n_i} \\
&= \sum_{j=i+1}^{q} \frac{1}{w_1(\lambda) \cdots w_{n_j - n_i}(\lambda)} \times \underbrace{\frac{w_1(\lambda) \cdots w_{n_j}(\lambda)}{w_1(\lambda_j) \cdots w_{n_j}(\lambda_j)}}_{\leq 1} e_{n_j - n_i}
\end{aligned}$$

The second factor is smaller than 1 since $x \mapsto w_n(x)$ is increasing for all n and since $\lambda \in \Lambda_i$ with $i < j$, that is, $\lambda \leq \lambda_i < \lambda_j$, $j = i + 1, ..., q$. The whole sum vanishes again by the hypothesis and since $n_j - n_i \geq N >> 0$.

Understanding the basic criterion with an example

The condition (iii) of the criterion is trivial since we are working with the backward shift so any finite supported vector will vanish under enough iterations of B.

Finally, for condition (iv) we have

$$\begin{aligned}
\|T_\lambda^{n_i} S_{\lambda_i}^{n_i}(e_0) - e_0\| &= \left\| \frac{w_1(\lambda)\cdots w_{n_i}(\lambda)}{w_1(\lambda_j)\cdots w_{n_i}(\lambda_i)} e_0 - e_0 \right\| \\
&= \left| \exp\left(\sum_{k=1}^{n_i} \log\big(w_k(\lambda)\big) - \sum_{k=1}^{n_i} \log\big(w_k(\lambda_i)\big) \right) - 1 \right| \\
&= \Big| \exp\big(f_{n_i}(\lambda) - f_{n_i}(\lambda_i)\big) - 1 \Big| \\
&\le 2\Big| f_{n_i}(\lambda) - f_{n_i}(\lambda_i) \Big| \\
&\le 2n_i |\lambda - \lambda_i| \\
&\le 2n_i \frac{\tau}{n_i} = 2\tau.
\end{aligned}$$

This can be made small by controlling τ.

The families of operators that interest us

Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$, and $\mathcal{D} = c_{00}(\mathbb{N})$.

For $K \subset (0, +\infty)^2$ compact, we write elements $\lambda \in K$ in the form $\lambda = (x, y)$. We consider a continuous family $\big(w(x)\big)_{x>0}$, where each $w(x) = \big(w_n(x)\big)_{n\in\mathbb{N}}$ is an **admissible** weight (in the sense that $B_{w(x)}$ is continuous on X).

We then define, for each $\lambda \in K$,

$$T_\lambda = B_{w(x)} \times B_{w(y)} \quad \text{and} \quad S_\lambda = F_{w(x)^{-1}} \times F_{w(x)^{-1}},$$

both acting on $X \times X$.

We are interested on weights that behave like

$$w_1(x) \cdots w_n(x) = \exp(x n^\alpha),$$

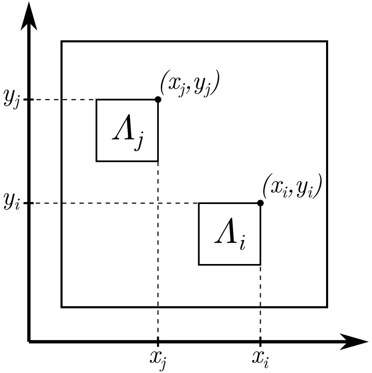
for some $\alpha \in (0, 1]$. In this case, the functions $x \overset{f_n}{\longmapsto} \sum_{k=1}^{n} \log\big(w_k(x)\big)$ are n^α-Lipschitz.

The families of operators that interest us

Looking at the Basic Criterion, we need to calculate terms of the form

$$T_{\lambda}^{n_i} S_{\lambda_j}^{n_j}(e_0, e_0) = \Big(B_{w(x)}^{n_i} F_{w(x_j)^{-1}}^{n_j} e_0, B_{w(y)}^{n_i} F_{w(y_j)^{-1}}^{n_j} e_0 \Big),$$

where $\lambda = (x, y) \in \Lambda_i$, $\lambda_j = (x_j, y_j) \in \Lambda_j$ and $j > i$.



The families of operators that interest us

For the first coordinate, we have

$$\begin{aligned} B^{n_i}_{w(x)} F^{n_j}_{w(x_j)^{-1}} e_0 &= \frac{w_{n_j-n_i+1}(x)\cdots w_{n_j}(x)}{w_1(x_j)\cdots w_{n_j}(x_j)} e_{n_j-n_i} \\ &= \frac{1}{w_1(x)\cdots w_{n_j-n_i}(x)} \times \frac{w_1(x)\cdots w_{n_j}(x)}{w_1(x_j)\cdots w_{n_j}(x_j)} e_{n_j-n_i} \\ &= \frac{1}{\exp(x(n_j-n_i)^\alpha)} \times \frac{\exp(x n_j^\alpha)}{\exp(x_j n_j^\alpha)} e_{n_j-n_i} \\ &= \frac{e_{n_j-n_i}}{\exp\left((x_j-x)n_j^\alpha + x(n_j-n_i)^\alpha\right)} \end{aligned}$$

Similarly, for the second coordinate we have

$$B^{n_i}_{w(y)} F^{n_j}_{w(y_j)^{-1}} e_0 = \frac{e_{n_j-n_i}}{\exp\left((y_j-y)n_j^\alpha + y(n_j-n_i)^\alpha\right)}$$

Hence, the conditions that link the covering to the sequence $(n_k)_k$ are

$$(x_j-x)n_j^\alpha + x(n_j-n_i)^\alpha \gg 0 \quad \text{and} \quad (y_j-y)n_j^\alpha + y(n_j-n_i)^\alpha \gg 0.$$

The families of operators that interest us

Now, for property (iv) of the Basic Criterion we make calculations similar to before with $j = i$.

$$T_{\lambda}^{n_i} S_{\lambda_i}^{n_i}(e_0, e_0) - (e_0, e_0) = \Big(B_{w(x)}^{n_i} F_{w(x_i)^{-1}}^{n_i} e_0 - e_0, B_{w(y)}^{n_i} F_{w(y_i)^{-1}}^{n_i} e_0 - e_0 \Big).$$

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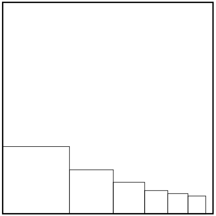
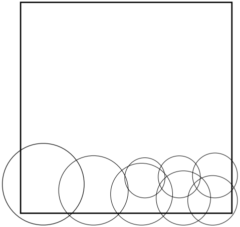
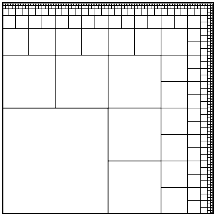
$$\begin{aligned}
\|B_{w(x)}^{n_i} F_{w(x_i)^{-1}}^{n_i} e_0 - e_0\| &= \left\| \frac{w_1(x) \cdots w_{n_i}(x)}{w_1(x_i) \cdots w_{n_i}(x_i)} e_0 - e_0 \right\| \\
&= \left| \exp \left(\sum_{k=1}^{n_i} \log \big(w_k(x)\big) - \sum_{k=1}^{n_i} \log \big(w_k(x_i)\big) \right) - 1 \right| \\
&= \Big| \exp \big(f_{n_i}(x) - f_{n_i}(x_i)\big) - 1 \Big| \\
&\leq 2|f_{n_i}(x) - f_{n_i}(x_i)| \\
&\leq 2n_i^{\alpha} |x - x_i|.
\end{aligned}$$

We would like to construct a partition fine enough so that $|x - x_i| < \frac{\tau}{n_i^{\alpha}}$. An analogous condition appears for the second coordinate.

How to get a partition in two dimensions?







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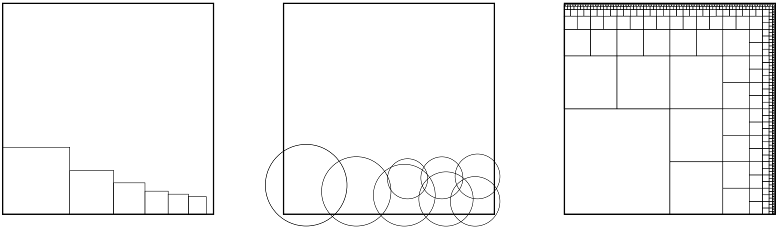


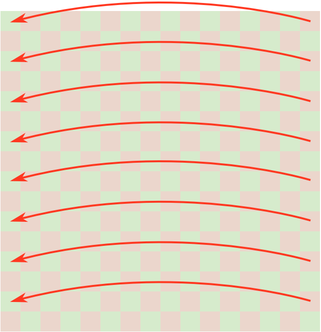
We can also consider a canonical dyadic covering, that is, by 4^m squares of side $1/2^m$. However, there are many ways of ordering this covering.

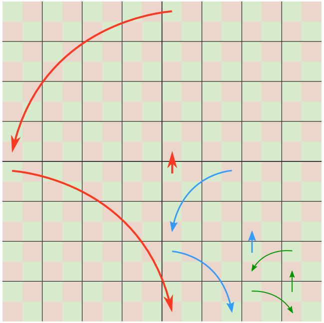
Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}
Λ_9	Λ_{10}	Λ_{11}	Λ_{12}
Λ_5	Λ_6	Λ_7	Λ_8
Λ_1	Λ_2	Λ_3	Λ_4

Λ_{16}	Λ_{15}	Λ_{14}	Λ_{13}
Λ_9	Λ_{10}	Λ_{11}	Λ_{12}
Λ_8	Λ_7	Λ_6	Λ_5
Λ_1	Λ_2	Λ_3	Λ_4

Λ_{16}	Λ_{15}	Λ_{12}	Λ_{11}
Λ_{13}	Λ_{14}	Λ_9	Λ_{10}
Λ_4	Λ_3	Λ_8	Λ_7
Λ_1	Λ_2	Λ_5	Λ_6

Backward jumps comparison (drawing for $m = 4$)





- 2^{m-1} BJ of size $1/2 - 1/2^m$
- No other BJ

- 3 BJ of size $1/2^2 + 1/2^m$
- 3 BJ of size $1/2^4 + 1/2^m$
- $\vdots$
- 3 BJ of size $1/2^{m-1} + 1/2^m$

BJ stands for "backward jump"

Steps of the construction

- Divide our square by 4^m smaller squares of side $1/2^m$;
- Choose a "fractal" ordering;
- Codify position of sub-squares in relation to the ordering chosen;
  - Write indexes in base 4;
  - Associate the size of the jump by distinguishing the base 4 entries of the indexes;
- associate a sequence that is just big enough for some properties to hold:
  - We manipulate $(x_j - x)n_j^\alpha + x(n_j - n_i)^\alpha \geq 0$ into a condition like $\|\lambda_j - \lambda_i\| \leq \frac{(n_j - n_i)^\alpha}{n_j^\alpha}$.
  - From $\Lambda_{4^m} \subset \left[x_{4^m}, x_{4^m} + \frac{\tau}{n_{4^m}^\alpha}\right] \times \left[y_{4^m}, y_{4^m} + \frac{\tau}{n_{4^m}^\alpha}\right]$ we need that $n_{4^m}^\alpha \leq \tau 2^m$.

The frontier between possibility and impossibility

For the weights given by $w_1(x)\cdots w_(x) = \exp(xn^\alpha)$, we know the following $\big($Bayart, CJr, Menet (2022)$\big)$:

- $\alpha < 1/2 \implies \big(B_{w(x)} \times B_{w(y)}\big)_{x,y>0}$ is common hypercyclic;
- $\alpha > 1/2 \implies \big(B_{w(x)} \times B_{w(y)}\big)_{x,y>0}$ is not common hypercyclic.

Open question: what can we say when $\alpha = 1/2$?

A satisfying criterion

Context: $d \geq 1$, $K \subset \mathbb{R}^d$ is compact, X is a Banach space, $\mathcal{D} \subset X$ is dense, $(T_\lambda)_{\lambda \in K}$ is a continuous family of operators on X with partial right inverses $(S_\lambda)_{\lambda \in K}$ on $\mathcal{D}$ (i.e. $T_\lambda S_\lambda(u) = u$ for all $u \in \mathcal{D}$).

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Theorem (Bayart, CJr, Menet (2022))

Suppose that there are $\alpha \in (0, 1/d)$, $\beta > \alpha d$ and $D > 0$ such that, for all $u \in \mathcal{D}$,

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(a) *there are $C, N > 0$ satisfying, for all $\lambda, \mu \in K$ and all $n \geq 0, k \geq N$ with $\|\lambda - \mu\|_\infty \leq D \frac{k^\alpha}{(n+k)^\alpha}$,*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq \frac{C}{k^\beta} \qquad \text{and} \qquad \left\|T_\lambda^{n} S_\mu^{n+k} u\right\| \leq \frac{C}{k^\beta}.$$

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(b) *for all $\varepsilon > 0$, there is $\tau > 0$ such that, for all $n \geq 1$ and $\lambda, \mu \in \Lambda$,*

$$\|\lambda - \mu\|_\infty \leq \frac{\tau}{n^\alpha} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \varepsilon.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

Applications to $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_n)})_{\lambda \in I^d}$

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Corollary

Let $\alpha \in (0, 1/d)$ *and let* $(w(x))_{x \in I}$ *be continuous and positive. Assume that there exist* $C_1, C_2, C_3 > 0$ *and* $N \geq 0$ *such that, for all* $n \geq N$,

- $x \in I \mapsto \sum_{j=1}^{n} \log(w_j(x))$ *is* $C_1 n^\alpha$*-Lipschitz;*
- $\inf_{x \in I} w_1(x) \cdots w_n(x) \geq C_2 \exp(C_3 n^\alpha)$.

Then $\bigcap_{\lambda \in I^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \varnothing$.

Applications to $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_n)})_{\lambda \in I^d}$

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Corollary

Let $(w(x))_{x \in I}$ be continuous and positive. Assume that there exist $C_1, C_2, \kappa > 0$ and $N \geq 0$ such that, for all $n \geq N$,

- $x \in I \mapsto \sum_{j=1}^{n} \log(w_j(x))$ *is $C_1 \log(n)$-Lipschitz;*
- $\inf_{x \in I} w_1(x) \cdots w_n(x) \geq C_2 n^\kappa$.

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Then $\bigcap_{\lambda\in I^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \varnothing$.

Example

Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$. Let Λ be a homogeneous Cantor subset of $(0, +\infty)^2$ with dissection ratio $\rho \in (0, 1/4)$. Then $\bigcap_{(a,b)\in\Lambda} HC(e^a B \times e^b B) \neq \varnothing$.

What's next?

- Can we optimize our partition in order to include the limit case $\alpha = 1/d$?
- Is there any more economical way of enumerating a dyadic partition?
- Is there a non-dyadic way of partitioning a square for common hypercyclicity purposes?
- Is the limit case actually false?

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